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Constructing Best Approximations on a Jordan Curve

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It is shown, within Bishop's constructive mathematics, that if a point is sufficiently close to a differentiable Jordan curve with suitably restricted curvature, then that point has a unique closest point on the curve.

1. INTRODUCTION

Although it seems intuitively clear that if J is a differentiable Jordan curve whose curvature is bounded away from zero, then there is a neighbourhood of J within which any point has a unique closest point on the curve, we have been unable to find any reference to such a result in the literature. In this paper we not only justify that intuition, but we do so constructively, using only intuitionistic logic.[1] It follows, in particular, that all our proofs can be translated into proofs within the context of recursive function theory [1, 7], Weihrauch's TTE [11, 12], and many other formal systems for computable analysis.

Our interest in the result arose from a constructive study of weak solutions of the Dirichlet Problem, where, in the classical theory, a condition related to the approximation property of Jordan curves is used to establish certain estimates for Sobolev spaces over smooth compact manifolds in $\mathbf{R}^N$; see [10; 8, Sect. 5.5]. Finding sufficient conditions for this property to hold constructively for a smooth compact manifold in $\mathbf{R}^N$ would require us to develop a constructive theory of manifolds, and, in view of the difficulties involved in the special case of a Jordan curve, appears to be an exceedingly difficult problem.

We assume that the reader has access to [3, 6, 9] for background information about constructive mathematics; additional references for constructive approximation theory include [4, 5].

[1] It appears that in order to establish our theorem with classical logic, one still needs something like Lemma 8 below, which requires a considerable amount of work; see later in the paper.

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By the *plane* we mean either $\mathbf{C}$ or $\mathbf{R}^2$, which we identify with each other in the usual way. We denote by $B(a, r)$ (respectively, $\overline{B}(a, r)$) the open (respectively, closed) ball with centre a and radius r in the plane. If $S \subset \mathbf{C}$ and the distance

$$\rho(z, S) \equiv \inf\{|z - s| : s \in S\}$$

exists, then we say that S is *located*. The *complement* of S in X is the set

$$X \sim S \equiv \{z \in X : \forall s \in S(z \neq s)\},$$

where $z \neq s$ means $|z - s| > 0$. We denote by $\operatorname{diam}(S)$ the diameter of the set S, when that diameter exists (which it does if, for example, S is totally bounded).

By a *Jordan curve* we mean a one-one, uniformly continuous mapping $f : \mathbf{T} \to \mathbf{R}^2$ with uniformly continuous inverse, where $\mathbf{T}$ is the unit circle in $\mathbf{R}^2$. We then identify f with its range J in the plane and with the mapping $\theta \mapsto f(\mathrm{e}^{\mathrm{i}\theta})$ of $[0, 2\pi)$ onto J. We say that $z_1 \equiv f(\mathrm{e}^{\mathrm{i}\theta_1})$ *precedes* $z_2 \equiv f(\mathrm{e}^{\mathrm{i}\theta_2})$ *on* J if $\theta_1 \leqslant \theta_2$, and that $f(\mathrm{e}^{\mathrm{i}\theta})$ is *between* z_1 and z_2 if $\theta_1 < \theta < \theta_2$; we denote the *arc* $\{f(\mathrm{e}^{\mathrm{i}\theta}) : \theta_1 \leqslant \theta \leqslant \theta_2\}$ of J joining z_1 and z_2 (in that order) by $J(z_1, z_2)$, and the length of that arc, if it exists, by $|J(z_1, z_2)|$. We say that J is *differentiable* if the mapping $\theta \mapsto f(\mathrm{e}^{\mathrm{i}\theta})$ is uniformly differentiable on each compact interval $I \subset \mathbf{R}$ with length less than 2π; in that case, for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 \leqslant \theta_2 - \theta_1 < 2\pi$ and $|\mathrm{e}^{\mathrm{i}\theta_1} - \mathrm{e}^{\mathrm{i}\theta_2}| < \delta$, then

$$|J(\mathrm{e}^{\mathrm{i}\theta_1}, \mathrm{e}^{\mathrm{i}\theta_2})| \equiv \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{\mathrm{d}f_1}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}f_2}{\mathrm{d}\theta}\right)^2}\, \mathrm{d}\theta < \varepsilon,$$

where $f = (f_1, f_2)$.

The *Jordan curve theorem* states, roughly, that the set of points u such that $\rho(u, J) > 0$ is the union of two components, the inside and the outside of J. If u belongs to the inside of J and v to the outside, we say that u and v are *on opposite sides of* J. For details of the Jordan curve theorem and its proof see [2].

A Jordan curve J is said to satisfy the *twin tangent ball condition* if there exists $R > 0$ such that for each $z \in J$ there exist points a_z, b_z on opposite sides of J, with

$$\overline{B}(a_z, R) \cap J = \{z\} = \overline{B}(b_z, R) \cap J.$$

It is straightforward to show that if J has continuous curvature, then the twin tangent ball condition implies that the radius of curvature of J at any point is at least R.

Our aim in this paper is to prove the following approximation theorem.

THEOREM. *Let J be a differentiable Jordan curve that satisfies the twin tangent ball condition.*[2] *Then there exists $r>0$ such that any point u of the plane that lies within r of J has a unique closest point on J; more precisely, if $\rho(u, J)<r$, then there exists $v \in J$ such that $|u-v|<|u-z|$ for all $z \in J \sim \{v\}$.*

Note that, for a given point u of the plane, it is a serious constructive problem to establish even the existence of a point $v \in J$ such that $|u-v| = \rho(u, J)$. We cannot appeal to the classical theorem that a continuous, real-valued function on a compact set attains its minimum, since there is a recursive example showing that it is essentially nonconstructive; see [6, Chap. 6].

The proof of our theorem depends on a series of lemmas, which we develop in the next section. The proofs of several of these lemmas are quite involved, so the reader is advised, on a first reading, to jump from here to the proof of the theorem itself, in Section 3.

2. PRELIMINARY RESULTS

Throughout this section, J is a Jordan curve satisfying the hypotheses of our theorem. We begin with two elementary, though nontrivial, lemmas in plane Euclidean geometry. The reader may find it helpful to draw diagrams to illustrate these lemmas.

We denote by $\overline{z_1 z_2}$ the line joining the two distinct points z_1, z_2 of the plane. By the *inclination* of two intersecting lines we mean the smallest angle between those lines.

LEMMA 1. *For $i=0,1,2$ let c_i, c_i' be points in the plane such that $|c_i - c_i'| = 2R > 0$, and let $z_i = \frac{1}{2}(c_i + c_i')$. There exists $t>0$ with the following property: if*

- $\min\{|z_i - c_0|, |z_i - c_0'|\} > R$ *for* $i \in \{1, 2\}$,
- $z_i \neq z_2$,
- $\overline{z_1 z_2}$ *is parallel to* $\overline{c_0 c_0'}$, *and*
- $\max\{|z_1 - z_0|, |z_2 - z_0|\} < t$,

then there exist distinct i, j such that

$$\text{either } B(c_i, R) \text{ intersects both } B(c_j, R) \text{ and } B(c_j', R)$$

$$\text{or else } B(c_i', R) \text{ intersects both } B(c_j, R) \text{ and } B(c_j', R).$$

[2] We doubt that the twin tangent ball condition is a necessary condition for our theorem, but we do not know if some weaker hypothesis will suffice.

Proof. Write $z_k = (x_k, y_k)$. We begin with two elementary geometric observations.

(a) If $z_0 = z_1 = 0$, $\overline{c_0 c_0'}$ is the imaginary axis, $0 < \theta < \pi/2$, and the inclination of $\overline{c_1 c_1'}$ to the imaginary axis is θ, then

$$\max\{|c_1 - c_0|, |c_1 - c_0'|\} < 2R \cos\frac{\theta}{2}$$

and

$$\max\{|c_1' - c_0|, |c_1' - c_0'|\} < 2R \cos\frac{\theta}{2}.$$

(b) If $z_1 = 0$, $x_2 = 0$, $|y_2| < 3R/2$, and the inclinations of $\overline{c_1 c_1'}$ and $\overline{c_2 c_2'}$ to the imaginary axis are at most

$$\alpha \equiv \cos^{-1}\left(\tfrac{3}{4}\right),$$

then either $B(c_1, R)$ intersects both $B(c_2, R)$ and $B(c_2', R)$ or $B(c_1', R)$ intersects both $B(c_2, R)$ and $B(c_2', R)$.

By observation (a), if $z_1 = z_0 = 0$ and the inclination of $\overline{c_1 c_1'}$ to the imaginary axis is greater than $\alpha/2$, then

$$\max\{|c_1 - c_0|, |c_1 - c_0'|\} < 2R - \varepsilon, \tag{1}$$

$$\max\{|c_1' - c_0|, |c_1' - c_0'|\} < 2R - \varepsilon, \tag{2}$$

where $\varepsilon = 2R(1 - \cos(\alpha/4))$. By continuity, there exists $t > 0$ such that if $z_0 = 0$, $|z_1| < t$, and $|\theta - \pi/2| > \alpha/2$, then (1) and (2) hold.

Now consider points z_k satisfying the bulleted conditions of the statement of the lemma. For convenience, we may assume that $z_0 = 0$, $c_0 = R$, and $c_0' = -R$, so that $x_1 = x_2$. Either the inclinations of $\overline{c_1 c_1'}$ and $\overline{c_2 c_2'}$ to the imaginary axis are less than α, or else the inclination of one of these two lines, say $\overline{c_1 c_1'}$, is greater than $\alpha/2$. In the first case, it follows from observation (b) that either $B(c_1, R)$ intersects both $B(c_2, R)$ and $B(c_2', R)$ or else $B(c_1', R)$ intersects both $B(c_2, R)$ and $B(c_2', R)$. In the second case, (1) holds and so $B(c_1, R)$ intersects both $B(c_0, R)$ and $B(c_0', R)$. ∎

LEMMA 2. *Let B_1, B_2 be two closed balls of radius R that are tangent at z. Let ζ, ζ' be points of the plane that lie outside B_1 and B_2, and on opposite sides of the line joining the centres of B_1 and B_2, such that*

$$\max\{|\zeta - z|, |\zeta' - z|\} < R. \tag{*}$$

If $0<r<R$, *then* z *lies in the interior of any circle of radius* r *that passes through both* ζ *and* ζ', *and hence in the interior of any ball of radius* r *that contains* ζ *and* ζ'.

Proof. We begin with another elementary geometrical observation.

> If A, B, C are vertices of a nondegenerate triangle such that the sum θ of the angles $A\hat{B}C$ and $A\hat{C}B$ is less than $\pi/2$ (radians), then the radius of the unique circle that passes through A, B, C is $|BC|/2\sin\theta$.

To prove the lemma, we may assume that $z=0$ and that the centres of B_1, B_2 are $(0,-R)$, $(0,R)$, respectively. Let w be the unique point in which the imaginary axis meets the segment $[\zeta,\zeta']$, let $0<r<R$, and let ζ,ζ' lie on the circle with centre ζ_0 and radius r. Choose $\delta>0$ such that $B(w,\delta)\subset B(\zeta_0,r)$. Either $|w|<\delta$, in which case $0\in B(\zeta_0,r)$, or else $|w|>0$. In the latter case, take, for example, $\mathrm{Im}(w)<0$. In view of $(*)$, we see that $-R<\mathrm{Im}(w)<0$ and that the interior of the segment $[\zeta,\zeta']$ meets the boundary of B_1 in two points ζ_1,ζ_1', where ζ_1 is between ζ and ζ_1'. Let $A\equiv(0,a)$ and $B\equiv(0,b)$ be the two points in which the boundary of $B(\zeta_0,r)$ meets the imaginary axis, where $a>\mathrm{Im}(w)>b$. Let θ be the sum of the angles $A\hat{\zeta}\zeta'$ and $A\hat{\zeta}'\zeta$; and ϕ the sum of the angles $0\hat{\zeta}_1\zeta_1'$ and $0\widehat{\zeta_1'}\zeta_1$. Noting that $|\zeta_1-\zeta_1'|<|\zeta-\zeta'|$, choose $\varepsilon>0$ such that if $\mathrm{Im}(w)<a<\varepsilon$, then

$$\frac{\sin\theta}{\sin\phi}<\frac{|\zeta-\zeta'|}{|\zeta_1-\zeta_1'|}.$$

Since, by the observation at the beginning of the proof,

$$\frac{|\zeta-\zeta'|}{2\sin\theta}=r<R=\frac{|\zeta_1-\zeta_1'|}{2\sin\phi},$$

we must have $a\geqslant\varepsilon$. Thus 0 is in the interior of the segment $[a,b]$ and is therefore in $B(\zeta_0,r)$. ▮

Before applying Lemma 1, we note that, although the full intermediate value theorem does not hold constructively (see [3, p. 8]), there are several useful constructive versions of that classical theorem, including the following one:

(IVT) *Let* $f:[0,1]\to\mathbf{R}$ *be a continuous function with* $f(0)<f(1)$. *There exists a sequence* (y_n) *in* $[f(0),f(1)]$ *such that if* $f(0)\leqslant y\leqslant f(1)$ *and* $y\neq y_n$ *for each* n, *then there exists* $x\in[0,1]$ *with* $f(x)=y$ [3, p. 63, Exercise 14].

LEMMA 3. *Let x, y, z be points of J such that z lies between x and y on J, and suppose that $[x, y]$ is bounded away from the line joining a_z and b_z. Then $|J(x, y)| \geqslant \operatorname{diam}(J(x, y)) > R/2$.*

Proof. It is clear that $|J(x, y)| \geqslant \operatorname{diam}(J(x, y))$. We may assume that $z = 0$, $a_z = -R$, and $b_z = R$. Since

$$0 < s \equiv \inf\{|\xi - \zeta| : \xi \in [x, y], \operatorname{Re}(\zeta) = 0\},$$

we may further assume that $[x, y]$ lies in the region $\operatorname{Re}(\zeta) > 0$. Either $\max\{|x|, |y|\} > R/2$ and therefore $\operatorname{diam}(J(x, y)) > R/2$, or else $\max\{|x|, |y|\} < R$. In the latter case, suppose that $J(x, y)$ does not intersect the region

$$D \equiv \{\zeta : \operatorname{Im}(\zeta) > R\} \cup \{\zeta : \operatorname{Im}(\zeta) < -R\}.$$

With t as in Lemma 1, we now use (IVT) to find $\lambda \in (0, t)$ such that there exists $z_1 = (x_1, y_1)$ between x and z on J with $|z_1| < t$ and $y_1 = \lambda$, and there exists $z_2 = (x_2, y_2)$ between z and y on J such that $|z_2| < t$ and $y_2 = \lambda$. Taking $z_0 = 0$ in Lemma 1, we see that one of the balls $B(a_{z_i}, R)$, $B(b_{z_i}, R)$ intersects both the balls $B(a_{z_j}, R)$, $B(b_{z_j}, R)$ and therefore intersects both the inside and the outside of J. Since this is absurd, we conclude that $J(x, y)$ intersects the region

$$\{\zeta : \operatorname{Im}(\zeta) > R/2\} \cup \{\zeta : \operatorname{Im}(\zeta) < -R/2\}$$

and hence that $\operatorname{diam}(J(x, y)) > R/2$. ∎

LEMMA 4. *For each $\alpha \in [0, \pi)$ there exists β with $0 < \beta < R$ such that if $0 < r \leqslant \beta$, $w \in \mathbf{R}^2$, $\alpha \leqslant \theta_1 \leqslant \theta_2 \leqslant 2\pi - \alpha$, and $|f(e^{i\theta_k}) - w| \leqslant r$ $(k = 1, 2)$, then $|f(e^{i\theta}) - w| < r$ for all θ in the open interval (θ_1, θ_2).*

Proof. We first observe that $f(e^{i\theta}) \mapsto \theta$ is a uniformly continuous mapping of $f([\alpha, 2\pi - \alpha])$ onto $[\alpha, 2\pi - \alpha]$. As J is differentiable, it follows that there exists β such that if $\alpha \leqslant \theta \leqslant \theta' < 2\pi - \alpha$ and $|f(e^{i\theta}) - f(e^{i\theta'})| < 2\beta$, then $|J(e^{i\theta}, e^{i\theta'})| < R/2$. Let w, r, θ_1, θ_2 be as in the hypotheses, and write $z_k \equiv f(e^{i\theta_k})$. Let $\theta_1 < \theta < \theta_2$ and $z = f(e^{i\theta})$; then $z \neq z_1, z_2$. Define

$$s \equiv \inf\{|\xi - \zeta| : \xi \in [z_1, z_2], \zeta \in M\},$$

where M is the line joining a_z and b_z. Then

$$|z_1 - z_2| \leqslant |z_1 - w| + |z_2 - w| \leqslant 2r \leqslant 2\beta,$$

so $|J(z_1, z_2)| < R/2$, by our choice of β, and it follows from Lemma 3 that $s = 0$. Moreover,

$$|z_1 - z| \leqslant |J(z_1, z)| \leqslant |J(z_1, z_2)| < R/2,$$

so as z_1 is distinct from z and lies outside the balls $B(a_z, R)$ and $B(b_z, R)$, it is a positive distance from M. Similarly, $|z_2 - z| < R/2$ and z_2 is a positive distance from M. Since $s = 0$, z_1 and z_2 lie on opposite sides of M; it follows from Lemma 2 that $|z - w| < r$. ∎

Let ω be the modulus of continuity for the mapping $\theta \mapsto f(e^{i\theta})$ on $\mathbf{R}$; so for each $\varepsilon > 0$, if $|f(e^{i\theta}) - f(e^{i\theta'})| > \varepsilon$, then $|\theta - \theta'| \geqslant \omega(\varepsilon)$. *In the remainder of this paper,* r_0 *will be the positive number* β *corresponding to* $\alpha = \omega(R/8)$ *in* Lemma 4.

LEMMA 5. *If* $\rho(u, J) < \min\{r_0, R/8\}$ *and* $|u - f(1)| > R/4$, *then for all but countably many* r *with*

$$\rho(u, J) < r < \min\{r_0, R/8\}, \tag{1}$$

there exist θ_1, θ_2 *such that* $0 < \theta_1 < \theta_2 < 2\pi$ *and*

$$\{\theta \in [0, 2\pi) : |f(e^{i\theta}) - u| \leqslant r\} = [\theta_1, \theta_2].$$

Proof. If $\theta \in [0, 2\pi)$ and

$$|f(e^{i\theta}) - u| \leqslant \frac{R}{8}, \tag{2}$$

then

$$|f(e^{i\theta}) - f(1)| \geqslant |u - f(1)| - |f(e^{i\theta}) - u| > \frac{R}{8}$$

and therefore $\alpha \leqslant \theta \leqslant 2\pi - \alpha$, where $\alpha = \omega(R/8)$. Since f is uniformly continuous on $[\alpha, 2\pi - \alpha]$, for all but countably many r with

$$\rho(u, J) < r < \min\{r_0, R/8\},$$

the set

$$\begin{aligned} S_r &\equiv \{\theta \in [0, 2\pi) : |f(e^{i\theta}) - u| \leqslant r\} \\ &= \{\theta \in [\alpha, 2\pi - \alpha] : |f(e^{i\theta}) - u| \leqslant r\} \end{aligned}$$

is compact. For such r, let $\theta_1 \equiv \inf S_r$ and $\theta_2 \equiv \sup S_r$. In view of (1) and (IVT), we can find $\theta, \theta' \in [0, 2\pi)$ such that

$$|f(e^{i\theta}) - u| < |f(e^{i\theta'}) - u| < r;$$

so $\theta_1 < \theta_2$. Using Lemma 4, we now see that $S_r = [\theta_1, \theta_2]$. ▮

If a, b are two distinct points of the plane, then the *ray from a towards b* is the set

$$\overrightarrow{ab} \equiv \{(1-t)\, a + tb : t \geqslant 0\}.$$

The proofs of the following two lemmas are simple exercises in geometry and trigonometry, and are omitted.

LEMMA 6. *Let z_1, z_2 be distinct points on the circle with centre w and radius $r_0 > 0$, let z be the midpoint of the minor arc joining z_1 and z_2, and let $t > 0$. Then there exists $\varepsilon > 0$ such that if $v \in \overrightarrow{zw}$ and $|v - z| > r_0 + t$, then $|v - z_1| > r_0 + \varepsilon$.*

LEMMA 7. *If C, C' are two circles of radius r that intersect in distinct points z_1, z_2 with $|z_1 - z_2| < \frac{4}{5} r$, and if the line joining the centres of the circles cuts C at z and C' at z', then $|z - z'| < \frac{1}{2} |z_1 - z_2|$.*

LEMMA 8. *Let z_1, z_2 be distinct points of J such that*

$$\gamma \equiv \max_{k=1,2} |u - z_k| < \tfrac{2}{5} r_0.$$

Then $\gamma > \rho(u, J)$.

Proof. Let $z_k = f(e^{i\theta_k})$, where $\theta_k \in [0, 2\pi)$, and assume without loss of generality that $\theta_1 < \theta_2$. Choose points w, w' on opposite sides of the line joining z_1 and z_2, such that

$$|w - z_k| = |w' - z_k| = r_0 \qquad (k = 1, 2).$$

Denote by C, C' the circles bounding $B(w, r_0)$ and $B(w', r_0)$, respectively. It follows from our choice of r_0 and Lemma 4 that for all $\theta \in (\theta_1, \theta_2)$,

$$\max\{|f(e^{i\theta}) - w|, |f(e^{i\theta}) - w'|\} < r_0. \tag{1}$$

Let z be the point in which $[w, w']$ intersects C. Since z is bounded away from z_1 and z_2, and, by (1), it is distinct from each point of $f((\theta_1, \theta_2))$, it follows by continuity that z is distinct from each point of $f([\theta_1, \theta_2])$. Now, this set is compact, since the mapping $\theta \mapsto f(\theta)$ is uniformly continuous on

$\mathbf{R}$ and the mapping $f(\theta) \mapsto \theta$ is uniformly continuous on $f([\theta_1, \theta_2])$. It follows from [3, Chap. 4, Lemma (3.8)] that z is bounded away from $f([\theta_1, \theta_2])$. Similarly, the point z' in which $[w, w']$ intersects C' is bounded away from $f([\theta_1, \theta_2])$. Hence

$$0 < t \equiv \tfrac{1}{6} \min\{\gamma, \rho(z, f([\theta_1, \theta_2])), \rho(z', f([\theta_1, \theta_2]))\}.$$

Let L be the line joining w and w'. By (IVT), there exist $\theta \in (\theta_1, \theta_2)$ and $\zeta \in L$ such that $|f(e^{i\theta}) - \zeta| < t$. Then

$$|\zeta - z| \geqslant |z - f(e^{i\theta})| - |f(e^{i\theta}) - \zeta| > 5t, \tag{2}$$

so either $\zeta \in \overrightarrow{aw}$ or $\zeta \in \overrightarrow{bw'}$, where

$$a = z + 5t(w - z),$$

$$b = z + 5t(w' - z).$$

But if $\zeta \in \overrightarrow{bw'}$, then $B(\zeta, t)$ is disjoint from $B(w, r_0) \cap B(w', r_0)$, which is absurd since $f(\theta) \in B(w, r_0) \cap B(w', r_0)$. Hence $\zeta \in \overrightarrow{aw} \subset \overrightarrow{zw}$. A similar argument shows that $|\zeta - z'| > 5t$ and $\zeta \in \overrightarrow{z'w'}$.

Now,

$$|z_1 - z_2| \leqslant |u - z_1| + |u - z_2| < \tfrac{4}{5} r_0,$$

so, by Lemma 7,

$$0 < s \equiv \tfrac{1}{2}(\tfrac{1}{2}\,|z_1 - z_2| - |z - z'|).$$

Hence there exists ε as in Lemma 6 such that

$$0 < \varepsilon < \min\{t, s\}.$$

Either $\rho(u, L) > 0$, in which case $|u - z_1| \neq |u - z_2|$ and the desired conclusion readily follows, or else $\rho(u, L) < \varepsilon$. In the latter case we show that $|u - \zeta| < |u - z_1|$. To this end, choose $v \in L$ such that $|u - v| < \varepsilon$. Then either $v \in \overrightarrow{zw}$ or else $v \in \overrightarrow{z'w'}$. Suppose the first alternative obtains. Note that, in view of (2) and the fact that w, w' are on opposite sides of $\overline{z_1 z_2}$, z is on the minor arc of C joining z_1 and z_2. Thus if $|v - z| > r_0 + t$, then

$$\begin{aligned} |u - z_1| &\geqslant |v - z_1| - |u - v| \\ &> r_0 + \varepsilon - \varepsilon \\ &= r_0, \end{aligned}$$

a contradiction. Hence $|v-z| \leqslant r_0+t$. Now, either $|v-z|>r_0-2t$ or $|v-z|<r_0-t$. In the first case we have $|v-w|<2t$,

$$\begin{aligned}|u-z_1| &\geqslant |v-z_1|-|u-v| \\ &\geqslant |w-z_1|-|v-w|-\varepsilon \\ &> r_0-2t-t \\ &= r_0-3t.\end{aligned}$$

Hence

$$\begin{aligned}|u-\zeta| &\leqslant |v-\zeta|+|u-v| \\ &< |v-z|-|z-\zeta|+\varepsilon \\ &< r_0+t-5t+t \qquad \text{(by (2))} \\ &= r_0-3t \\ &< |u-z_1|.\end{aligned}$$

In the case $|v-z|<r_0-t$, either $|v-\zeta|<\gamma-2t$ and therefore

$$|u-\zeta|<\gamma-2t+\varepsilon<\gamma,$$

or else, as we may assume, $v \neq \zeta$. We now have two subcases to consider.

Subcase 1. v lies strictly between ζ and w on the ray $\overrightarrow{zw}$. Then

$$\begin{aligned}|v-z_1| &\geqslant |w-z_1|-|w-v| \\ &= r_0-(|w-z|-|z-\zeta|-|\zeta-v|) \\ &= |z-\zeta|+|\zeta-v| \\ &> 5t+|\zeta-v|,\end{aligned}$$

and therefore

$$\begin{aligned}|u-\zeta| &< |v-\zeta|+\varepsilon \\ &< |v-z_1|-5t+t \\ &< |u-z_1|+\varepsilon-4t \\ &< |u-z_1|-3t.\end{aligned}$$

Subcase 2. v lies strictly between z and ζ on the ray $\overrightarrow{zw}$. Then v, ζ lie on the interior of the segment $[z, z']$ and, by elementary geometry, $|v-z_1| \geqslant \frac{1}{2}|z_1-z_2|$; whence

$$\begin{aligned}|u-\zeta| &< |v-\zeta|+\varepsilon\\ &\leqslant |z-z'|+s\\ &= \tfrac{1}{2}|z_1-z_2|-s\\ &\leqslant |v-z_1|-s\\ &< |u-z_1|+\varepsilon-s\\ &< |u-z_1|.\end{aligned}$$

This completes the proof when $v \in \overrightarrow{zw}$. The proof when $v \in \overrightarrow{z'w'}$ is similar. ▮

It is now simple to give a classical proof of our main theorem. To this end, suppose that the hypotheses of the theorem hold, and let $\rho(u, J) < \frac{2}{5}r_0$. Choose $v \in J$ such that $|u-v| = \rho(u, J)$. (This is the first nonconstructive step in the proof.) If there exists $z \in J$ such that $z \neq v$ and $|u-z| = \rho(u, J)$, then Lemma 8 shows that

$$\rho(u, J) < \max\{|u-v|, |u-z|\},$$

which is absurd. It follows that $|u-v| < |u-z|$ for all $z \in J \sim \{v\}$. (This is also a nonconstructive step: ruling out the possibility that u has two distinct closest points on J does not suffice to establish, constructively, the strong uniqueness conclusion of the theorem.)

3. PROOF OF THE MAIN THEOREM

We are now able to prove our main theorem constructively. To this end, assume that the hypotheses of the theorem are satisfied. Consider $u \in \mathbf{R}^2$ such that

$$\rho(u, J) < r \equiv \min\left\{\frac{2}{5} r_0, \frac{R}{8}\right\}.$$

Since, by Lemma 3, $\operatorname{diam}(J) > R/2$, there exists $\phi \in [0, 2\pi)$ such that $|u - f(\mathrm{e}^{\mathrm{i}\phi})| > R/4$; replacing f by the mapping $\theta \mapsto f(\mathrm{e}^{\mathrm{i}(\phi-\theta)})$, we may assume that $|u - f(1)| > R/4$. Using Lemma 5, choose θ_1, θ_1', and r_1 such that

$$0 < \theta_1 < \theta_1' < 2\pi,$$

$$\rho(u, J) < r_1 < \min\left\{\frac{2}{5} r_0, \frac{R}{8}, \rho(u, J) + 1\right\},$$

and

$$S_1 \equiv \{\theta \in [0, 2\pi) : |f(e^{i\theta}) - u| \leqslant r_1\} = [\theta_1, \theta_1'].$$

Suppose that, for some $n \geqslant 1$, we have constructed θ_n, θ_n', and r_n such that

(1) $0 < \theta_n < \theta_n' < 2\pi$,
(2) $\rho(u, J) < r_n < \min\{\frac{2}{5}r_0, r_{n-1}, \rho(u, J) + (1/n)\}$,
(3) $S_n \equiv \{\theta \in [0, 2\pi) : |f(e^{i\theta}) - u| \leqslant r_n\} = [\theta_n, \theta_n']$, and
(4) $\theta_n' - \theta_n \leqslant (\frac{2}{3})^{n-1} (\theta_1' - \theta_1)$.

Let

$$z_1 = f(e^{i((1/3)\,\theta_n + (2/3)\,\theta_n')}),$$
$$z_2 = f(e^{i((2/3)\,\theta_n + (1/3)\,\theta_n')}).$$

Writing

$$\gamma \equiv \max\{|u - z_1|, |u - z_2|\},$$

we see from properties (2) and (3) that $\gamma \leqslant r_n < \frac{2}{5}r_0$; whence, by Lemma 8, $\rho(u, J) < \gamma$. Using Lemma 5 again, we can now find r_{n+1}, θ_{n+1}, and θ_{n+1}' such that $0 < \theta_{n+1} < \theta_{n+1}' < 2\pi$,

$$\rho(u, J) < r_{n+1} < \min\left\{r_n, \gamma, \rho(u, J) + \frac{1}{n+1}\right\}$$

and

$$S_{n+1} \equiv \{\theta \in [0, 2\pi) : |f(e^{i\theta}) - u| \leqslant r_{n+1}\} = [\theta_{n+1}, \theta_{n+1}'].$$

This completes the inductive construction of sequences (θ_n), (θ_n'), and (r_n) satisfying properties (1)–(4).

Now, (S_n) is a descending sequence of compact intervals, and, by property (4), the length of S_n converges to 0. Hence $\bigcap_{n=1}^{\infty} S_n$ consists of a single point θ_∞. It follows from properties (2) and (3) that $|u - v| = \rho(u, J)$, where $v = f(e^{i\theta_\infty})$. If $z \in J \sim \{v\}$, then either $|u - z| > \rho(u, J)$ or else $|u - z| < \frac{2}{5}r_0$; in the latter case we see from Lemma 8 that

$$\max\{|u - v|, |u - z|\} > \rho(u, J)$$

and therefore that $|u - z| > \rho(u, J)$. ∎

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